

SOME LOWER ESTIMATES OF ADM MASS AND BROWN-YORK MASS

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ABSTRACT. We give some lower estimates of the ADM mass of an asymptotically flat (AF) Riemannian manifold without assuming that the scalar curvature of the manifold is nonnegative. Some sufficient conditions for an AF manifold to have nonnegative ADM mass are obtained. We also give some lower estimates of the Brown-York mass of a compact three manifold with smooth boundary. From these estimates, we generalize some previous results of the authors.

§1 Introduction.

More than twenty years ago, Schoen and Yau [SY1-2] proved the positive mass theorem. Later, using spinors Witten [W] gave a simple proof of the result. A mathematical rigorous proof of Witten's argument was given by Parker and Taubes [PT], see also [B1]. For the time-symmetric case, the positive mass theorem asserts that the Arnowitt-Deser-Misner (ADM) mass of each end of a three dimensional asymptotically flat (AF) manifold M with finitely many ends with L^1 integrable and nonnegative scalar curvature is nonnegative. Moreover, if the ADM mass of one of the ends is zero then the manifold has only one end and is isometric to the three dimensional Euclidean space. In the time-symmetric case, the scalar curvature is nonnegative means physically that the local mass density is nonnegative. The condition that the scalar curvature being in L^1 is necessary in order that the ADM mass is well-defined, see [B1].

In [ZZ], L. Zhang and X. Zhang studied an interesting question asked by S.-T. Yau how the condition that the scalar curvature is nonnegative can be relaxed so that the ADM mass of an AF manifold is still nonnegative. In [ZZ], they proved that the positive mass theorem is still true under the assumptions that the first Dirichlet eigenvalue and the first eigenvalue of Neumann type of the conformal

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Laplacian operator are nonnegative. Motivated by the above mentioned results, in this work we shall discuss lower bounds of the ADM mass of an AF manifold without assuming that the scalar curvature is nonnegative. See Theorems 3.1 and 3.2 for more details. From these results, we obtain conditions in terms of the geometry of the AF manifold so that the ADM mass of the manifold is nonnegative. More precisely, we have the following (Corollaries 3.1 and 3.3):

Theorem 1.1. *Let (M^3, g) be an AF manifold with one end such that its scalar curvature \mathcal{R} is in $L^1(M)$. Let \mathcal{R}_+ and \mathcal{R}_- be positive part and negative part of \mathcal{R} respectively. Suppose one of the following is true, then the ADM mass of M is nonnegative.*

$$(a) \quad \left(\int_M \left(\frac{\mathcal{R}_-}{4} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} < \frac{\Lambda}{2}, \text{ and}$$

$$\left[\Lambda - 2 \left(\int_M \left(\frac{\mathcal{R}_-}{4} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} \right] \int_M \mathcal{R}_+ \geq \left[\Lambda + 2 \left(\int_M \left(\frac{\mathcal{R}_+}{4} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} \right] \int_M \mathcal{R}_-.$$

where Λ is the Sobolev constant of (M, g) .

$$(b) \quad \left(\int_M \left(\frac{\mathcal{R}_-}{4} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} < \Lambda, \text{ and}$$

$$B \geq C_1 A^2 \left[\sup_M |Rm| \int_M \mathcal{R}_- + \|\nabla Rm\|_2 \left(\int_M \mathcal{R}_- \right)^{\frac{1}{2}} \right]$$

where

$$B = \inf_D \left\{ \int_{M \setminus D} |Rm|^2 \mid \text{vol}(D) \leq C_2 \left[A \Lambda^{-1} \int_M \mathcal{R}_- \right]^3 \right\},$$

C_1 and C_2 are positive absolute constants, and A is a positive constant depending only on Λ and \mathcal{R}_- .

A condition similar to (a) can also be obtained from the proof of [ZZ]. Condition (a) implies that $\int_M \mathcal{R} \geq 0$. In fact the assumptions in [ZZ] that the first eigenvalue of Neumann type is nonnegative also imply that $\int_M \mathcal{R} \geq 0$. Hence if $\int_M \mathcal{R} < 0$, then we cannot apply (a) of the theorem.

In [BF, FK], Bray-Finster and Finster-Kath obtained lower estimates of the ADM mass in terms of the curvature under the assumption that the scalar curvature is nonnegative. Condition (b) in Theorem 1.1 is obtained by generalizing their results to the case that the scalar curvature may be negative somewhere. If \mathcal{R} is nonnegative, then $B \geq 0$, condition (b) is automatically satisfied.

In the second part of this work, we shall discuss similar issues for the Brown-York mass of a compact manifold with smooth boundary.

Let (Ω^3, g) be a 3-dimensional compact manifold with smooth boundary. For simplicity, in this paper we always assume that $\partial\Omega = \Sigma$ is connected. Suppose the Gauss curvature of Σ is positive, then by a classical result we know that Σ can be isometrically embedded into \mathbb{R}^3 . Let H_0 be the mean curvature of image of the embedding in \mathbb{R}^3 with respect to the outward norm. The Brown-York mass of (Ω, g) is defined as (see [BY 1-2]):

$$m_{BY}(\Omega) = \int_{\Sigma} (H_0 - H) d\sigma$$

where $d\sigma$ is the volume element of Σ , and H is the mean curvature of $\partial\Omega$ with respect to original metric g and outward norm. In our convention, the mean curvature of the unit sphere in \mathbb{R}^3 is 2. In [ST1], under the assumption that Σ has positive Gauss curvature, it was proved that if (i) the scalar curvature of (Ω, g) is nonnegative and (ii) $H > 0$, then $m_{BY}(\Omega) \geq 0$. Moreover, equality holds if and only if Ω is a domain in \mathbb{R}^3 . The condition on the Gauss curvature of Σ has been relaxed in [ST2] where the Gauss curvature of the boundary is only assumed to be nonnegative. In the second part of this paper, we shall try to relax condition (i) or condition (ii). As for ADM mass of an AF manifold, we shall first give some lower estimates of the Brown-York mass.

Consider a compact manifold (Ω^3, g) with smooth boundary Σ which has positive Gauss curvature. Let \mathcal{R} be the scalar curvature of M and let $s_0 > 0$ be such that $d(x, \partial\Omega)$ is smooth in $\Omega_{s_0} = \{x \mid 0 < d(x, \partial\Omega) \leq s_0\}$. We have the following (see Theorem 4.1 and Corollary 4.2):

Theorem 1.2. *With the above notations, suppose the mean curvature of the level set $\{x \mid d(x, \Sigma) = s\}$ is positive with respect to the outward normal for $0 \leq s \leq s_0$. Then there exists a constant $C > 0$ depending only on s_0 , H_{\min} , Λ and $|\Omega|$ where H is the mean curvature of Σ and Λ is the Sobolev constant of Ω such that if*

$$\sup_{\Omega} \mathcal{R}_- \leq C,$$

then the Brown-York mass of Ω satisfies:

(a)

$$(1.1) \quad m_{BY}(\Omega) \geq \frac{\Lambda - 2\beta}{32(\Lambda + \delta - \beta)} \left(\int_{\Omega} \mathcal{R}_+ - \frac{\Lambda + 2\delta}{\Lambda - 2\beta} \int_{\Omega} \mathcal{R}_- \right)$$

provided that $\beta = \left(\int_{\Omega} \frac{1}{8} \mathcal{R}_-^{\frac{3}{2}} \right)^{\frac{2}{3}} < \frac{\Lambda}{2}$ where $\delta = \left(\int_{\Omega} \frac{1}{8} \mathcal{R}_+^{\frac{3}{2}} \right)^{\frac{2}{3}}$; and

(b)

$$(1.2) \quad m_{BY}(\Omega) \geq \frac{\lambda}{4} \int_{\Omega} \frac{\mathcal{R}}{8\lambda + \mathcal{R}}$$

provided that $8\lambda + \mathcal{R} > 0$ in Ω where λ is the first Dirichlet eigenvalue of the Laplacian of Ω .

From the theorem one can conclude that the Brown-York mass is still nonnegative if the mean curvature is positive and the scalar curvature is not very negative. One may ask what would happen if the mean curvature is negative somewhere. In this case we have the following, see Theorem 4.2 and Corollary 4.3:

Theorem 1.3. *With the same notations as in Theorem 1.2, suppose $\mathcal{R} \geq 0$ and let $\xi = \frac{1}{4}\mathcal{R}_{\min}^{\frac{1}{2}}$ where $\mathcal{R}_{\min} = \inf_{\Omega_{s_0}} \mathcal{R}$. Let H be the mean curvature of $\{x \mid d(x, \partial\Omega) = s\}$, $0 \leq s \leq s_0$, $H_+ = \max\{H, 0\}$ and $H_{\min} = \min_{\partial\Omega} H$. Suppose*

- (i) $\xi \geq H_+ \tanh(\xi s_0)$ in Ω_{s_0} ; and
- (ii) $\xi \tanh(\xi s_0) \geq -4H_{\min}$.

Then

$$\mathfrak{m}_{\text{BY}}(\Omega) \geq \frac{1}{4}|\Sigma| \xi \tanh(\xi s_0).$$

Moreover, the $\mathfrak{m}_{\text{BY}}(\Omega)$ can also be estimated from below as in (1.1) and (1.2).

From Theorem 1.3, one can conclude that the Brown-York mass is still nonnegative if the scalar curvature is positive but the mean curvature is not too negative. The conditions of the theorem are satisfied for some s_0 if $H \geq 0$. In particular, (1.1) and (1.2) give lower bounds for the Brown-York mass for compact manifolds with nonnegative scalar curvature such that its boundary has positive Gauss curvature and positive mean curvature.

It seems likely that the Brown-York mass is nonnegative irrespectively to the sign of the mean curvature. For example, if the mean curvature is negative everywhere, then obviously the Brown-York mass is nonnegative provided the boundary has positive Gauss curvature. However, it is still unclear whether this is true in general.

The paper is organized as follows. In §2, we shall give conditions on the existence of asymptotically constant harmonic spinors on AF manifolds. The results will be used in the next section. In §3, we shall obtain different types of lower bounds of the ADM mass in terms of the geometry of the AF manifold. In §4, we shall relax the assumptions on the scalar curvature of a compact manifold or the mean curvature of its boundary and generalize some results in [ST1]. To do this we shall first obtain lower bounds of the Brown-York mass of a compact three manifold with smooth boundary. In §5, we shall construct examples which are related to the results in Theorem 1.1. We shall give some interesting applications of quasi-spherical metrics on the relation between the classical Minkowski inequalities for convex surfaces and the positive mass theorem of Herzlich [H].

§2 Construction of harmonic spinors.

In this section, we shall discuss conditions on an AF spin manifold (see the definition below) so that one can construct harmonic spinors which are asymptotically parallel near infinity. Here we do not assume that the scalar curvature is nonnegative. The construction will be used to give an expression for the ADM mass of the manifold. With the help of these, we shall estimate the lower bound of the ADM mass and obtain some conditions so that the ADM mass is nonnegative. For simplicity, we always assume the AF manifold has only one end.

A complete noncompact spin manifold (M^n, g) , $n \geq 3$, is said to be *asymptotically flat (AF)* if there is a compact set K and a diffeomorphism $\phi : \mathbb{R}^n \setminus B_R(0) \rightarrow M \setminus K$ for some Euclidean ball $B_R(0)$ with center at the origin, such that in the standard coordinates of \mathbb{R}^n ,

$$g_{ij} = \delta_{ij} + b_{ij}$$

with

$$(2.1) \quad \|b_{ij}\| + r\|\partial b_{ij}\| + r^2\|\partial\partial b_{ij}\| = O(r^{2-n})$$

where r and ∂ denote Euclidean distance and the standard gradient operator on \mathbb{R}^n and the norms are taken with respect to the Euclidean metric. Moreover, the scalar curvature \mathcal{R} of M is assumed to be in $L^1(M)$ so that the ADM mass of M is well-defined by [B1]. Here the ADM mass of M is given by

$$c(n)\mathfrak{m} = \lim_{r \rightarrow \infty} \int_{S(r)} (g_{ii,j} - g_{jj,i}) dS^i$$

where $S(r)$ is the Euclidean sphere of radius r , dS^i is the normal surface area of $S(r)$ and $c(n) > 0$ is a normalizing constant.

First, we need some results on the existence of positive solutions of equation of the form

$$(2.2) \quad Lu = \Delta u - qu = 0$$

where $q = O(r^{-n})$. Here and below, when we say a function $f = O(r^\alpha)$ we mean that $|f(x)| \leq C(1 + r(x))^\alpha$ for some constant C for all $x \in M$ and $r(x)$ is the geodesic distance of x from a fixed point.

Lemma 2.1. *There is a constant C depending only on M and q such that if u is a positive solution of (2.2), then for any $r > 0$,*

$$\sup_{B_{2r}(p) \setminus B_p(r)} u \leq C \inf_{B_{2r}(p) \setminus B_p(r)} u$$

for all $r > 0$, where p is a fixed point and $B_p(r)$ is the geodesic ball of radius r with center at p .

Proof. Since $q = O(r^{-n})$ and g_{ij} is uniformly equivalent to the Euclidean metric, the result follows from [GT, Theorem 8.20], see also [ZZ, p. 666].

Suppose the first Dirichlet eigenvalue of L in (2.2) is nonnegative, namely:

$$\int_M (|\nabla v|^2 + qv^2) \geq 0$$

for all $v \in C_0^\infty(M)$. By a well-known fact, the first Dirichlet eigenvalue of L on any compact domain of M is positive (see [FS, p.201]), we will use this fact from time to time. Then the following comparison theorem holds.

Lemma 2.2. *Assume that the first Dirichlet eigenvalue of L on M is nonnegative. Suppose u and v are two solutions of (2.2) in a bounded domain Ω such that $u \geq v$ on $\partial\Omega$. Then $u > v$ in Ω unless $u \equiv v$.*

Proof. Let $w = u - v$, then w is also a solution of (2.2) and $w \geq 0$ on $\partial\Omega$. Suppose $\inf_{\Omega} w < 0$. Let $D = \{w < 0\}$. Then D is an open subset of Ω and $w = 0$ on ∂D . We have

$$\int_D |\nabla w|^2 + qw^2 = 0.$$

Since the first Dirichlet eigenvalue of L in M is nonnegative, the first eigenvalue of L in D must be positive. This is impossible. Hence $\min_{\Omega} w \geq 0$. By the strong maximum principle [GT, p. 35] we conclude that $w > 0$ in Ω unless $w \equiv 0$.

It is known that the first Dirichlet eigenvalue of L is nonnegative if and only if (2.2) has a positive solution, see [FS, p.201]. We want to discuss the asymptotically behavior of the positive solution when it exists.

Lemma 2.3. *Let q be a smooth function in M such that $q = O(r^{-n})$ and let*

$$v(x) = - \int_M G(x, y) q(y) dy.$$

Then v is the unique solution of $\Delta v = q$ with $\lim_{x \rightarrow \infty} v(x) = 0$. Moreover, $v = O(r^{2-n})$ and $\int_M |\nabla v|^2 < \infty$.

Proof. The fact that v is the unique solution of $\Delta v = q$ with $\lim_{x \rightarrow \infty} v(x) = 0$ follows from the fact that M is AF and Lemma 2.1. The fact that $v = O(r^{2-n})$ also follows from the fact that M is AF. To prove that $|\nabla v| \in L^2(M)$, multiply $\Delta v = q$ by $\varphi^2 v$ and integrating by parts, where φ is a cutoff function, we have

$$\int_M \varphi^2 |\nabla v|^2 \leq 4 \int_M |\nabla \varphi|^2 v^2 + \int_M |qv| \varphi^2.$$

Using the fact that $v = O(r^{2-n})$, $q = O(r^{-n})$ it is easy to see that $\int_M |\nabla v|^2 < \infty$.

Lemma 2.4. *Suppose the first Dirichlet eigenvalue of L on M is nonnegative. Then the positive solution of (2.2) is unique in the sense that if u and v are two positive solutions of (2.2), then $u = \beta v$ for some $\beta > 0$. Moreover, any positive solution of (2.2) is bounded and $\lim_{x \rightarrow \infty} u(x)$ exists. In fact, $u(x) = b + O(r^{2-n})$ for some constant $b \geq 0$.*

Proof. Let u and v be two positive solutions of (2.2). We may assume that there exist $x_i \rightarrow \infty$ such that $u(x_i) \geq v(x_i)$. Suppose $r(x_i) = d(p, x_i) = R_i$ where p is a fixed point. By Lemma 2.1, there is a constant $C_1 > 0$ depending only on M and q such that

$$\inf_{\partial B_p(R_i)} u \geq C_1 u(x_i) \geq C_1 v(x_i) \geq C_1^2 \sup_{\partial B_p(R_i)} v.$$

By Lemma 2.2, we conclude that $u \geq C_1^2 v$ in $B_p(R_i)$ for all i . Hence $u \geq C_1^2 v$ in M . Let

$$\beta^* = \sup\{\beta > 0 \mid u \geq \beta v \text{ in } M\}.$$

Then $u - \beta^* v$ is a solution of (2.2). Suppose $u \not\equiv \beta^* v$, then $w = u - \beta^* v > 0$ in M by Lemma 2.2. Choose any $R'_i \rightarrow \infty$ and choose any $x'_i \in \partial B_p(R'_i)$. Let $a_i = \frac{w(x'_i)}{v(x'_i)}$. We claim that a_i is bounded from below away from zero. Otherwise, passing to a subsequence, we may assume that $a_i \rightarrow 0$. Since w is also a positive solution of (2.2), we can argue as before to conclude that

$$\inf_{\partial B_p(R'_i)} a_i v \geq C_1 a_i v(x'_i) = C_1 w(x'_i) \geq C_1^2 \sup_{\partial B_p(R'_i)} w.$$

By Lemma 2.2, we have that $a_i v \geq C^2 w$ in $B_p(R'_i)$. From this it is easy to see that $w \equiv 0$, which is a contradiction. Hence $a_i \geq a > 0$ for all i . Then as before

$$\inf_{\partial B_p(R'_i)} w \geq C_1 w(x'_i) \geq C_1 a v(x'_i) \geq C_1^2 \sup_{\partial B_p(R'_i)} v.$$

We conclude that $w \geq C_1^2 a v$ and $u \geq (\beta^* + C_1^2 a) v$. This contradicts the definition of β^* . So $u = \beta^* v$. This completes the proof of uniqueness.

It remains to prove the other assertions in the lemma. Suppose u is unbounded. Then by Lemma 2.1, there exist $r_i \rightarrow \infty$ such that $\inf_{\partial B_p(r_i)} u \geq i$. By Lemma 2.3, let φ be the solution of $\Delta \varphi = q$ such that $\varphi = O(r^{2-n})$. Then

$$\begin{aligned} \Delta \log u &= \frac{\Delta u}{u} - |\nabla \log u|^2 \\ &= q - |\nabla \log u|^2 \\ &\leq \Delta \varphi. \end{aligned}$$

Since $\inf_{\partial B_p(r_i)} \log u \geq \log i$, $\varphi \leq \log u - \log i + 1$ in $B_p(r_i)$ for all i provided i is large. This is impossible. Hence u is bounded.

Finally, $\Delta u = qu$ and $qu = O(r^{-n})$ because u is bounded. Let v' be the solution of $\Delta v' = qu$ given by Lemma 2.3. Then $\Delta(u - v') = 0$ and hence $u - v'$ must be constant by Lemma 2.1. Since $v' = O(r^{2-n})$, the lemma follows.

Next, we will give a necessary and sufficient condition that the solution in Lemma 2.4 has a positive limit at infinity.

Theorem 2.1. *Let q be a smooth function on M such that $q = O(r^{-n})$. Then*

$$\Delta u - qu = 0$$

has a positive solution u satisfying $u = 1 + O(r^{2-n})$ if and only if there is a smooth function $f \geq 0$, $f \not\equiv 0$ and $f = O(r^{-n})$ such that the operator $\Delta - q + f$ has nonnegative Dirichlet eigenvalue.

Proof. Suppose there is a smooth function $f \geq 0$, $f \not\equiv 0$ and $f = O(r^{-n})$ such that the operator $\Delta - q + f$ has nonnegative Dirichlet eigenvalue. Then the first Dirichlet eigenvalue of $\Delta - q$ is also nonnegative. By Lemma 2.4, we can find a positive solution of $Lu = 0$ such that $u = a + O(r^{2-n})$ for some $a \geq 0$. Suppose $a = 0$, then for any $0 < \epsilon < \max_M u$, the set $\{u > \epsilon\}$ is bounded in M and the family of sets $\{u > \epsilon\}$ with $\max_M u > \epsilon > 0$ exhausts M . Let $0 < \epsilon_0 < \max_M u$ and let $R > 0$ be fixed. Then for $0 < \epsilon \leq \epsilon_0$, $B_p(R) \subset \{u > \epsilon\}$ if ϵ is small enough, where $B_p(R)$ is the geodesic ball with center at a fixed point p with radius R . For such ϵ , we have

$$\begin{aligned}
0 &\leq \int_{\{u > \epsilon\}} [|\nabla(u - \epsilon)|^2 + q(u - \epsilon)^2 - f(u - \epsilon)^2] \\
&= -\epsilon \int_{\{u > \epsilon\}} q(u - \epsilon) - \int_{\{u > \epsilon\}} f(u - \epsilon)^2 \\
&= -\epsilon \int_{\{u > \epsilon\}} qu + \epsilon^2 \int_{\{u > \epsilon\}} q - \int_{\{u > \epsilon\}} f(u - \epsilon)^2 \\
&= -\epsilon \int_{\{u > \epsilon\}} qu + \epsilon^2 \int_{\{u > \epsilon\}} \frac{|\nabla u|^2}{u^2} + \epsilon \int_{\{u > \epsilon\}} \frac{\partial u}{\partial \nu} - \int_{\{u > \epsilon\}} f(u - \epsilon)^2 \\
&\leq -\epsilon \int_{\{u > \epsilon\}} qu + \int_{\{u > \epsilon\} \setminus B_p(R)} |\nabla u|^2 + \epsilon^2 \int_{B_p(R)} \frac{|\nabla u|^2}{u^2} - \int_{\{u > \epsilon_0\}} f(u - \epsilon)^2
\end{aligned}$$

where ν is the unit outward normal of $\{u = \epsilon\}$ so that $\frac{\partial u}{\partial \nu} \leq 0$ and we have used the fact that $f \geq 0$. Let $\epsilon \rightarrow 0$ using the fact that $qu = O(r^{2-2n})$ so that $qu \in L^1(M)$ we have

$$0 \leq \int_{M \setminus B_o(R)} |\nabla u|^2 - \int_{\{u > \epsilon_0\}} fu^2.$$

By Lemma 2.3, $\int_M |\nabla u|^2 < \infty$, hence if we let $R \rightarrow \infty$ and then let $\epsilon_0 \rightarrow 0$, we have

$$0 \leq - \int_M fu^2.$$

Since $u > 0$, $f \geq 0$ and $f \not\equiv 0$, this is impossible.

Conversely, suppose there is a positive solution of $\Delta u - qu = 0$ with $u = 1 + O(r^{2-n})$. Let q_+ be the positive part of q , then $q_+ = O(r^{-n})$. Let k be a smooth positive function such that $k = O(r^{-n})$ and $k > q_+$. It is easy to see that the first Dirichlet eigenvalue of $\Delta - k$ is nonnegative. By Lemma 2.4, there is bounded and positive solution w of $\Delta w - kw = 0$. By multiplying w by a positive number, we may assume that $u - w \geq a > 0$ for some a , where we have used the fact that $\inf_M u > 0$ and w is bounded. Let $f = (\Delta w - qw)/(u - w)$. Since $k = O(r^{-n})$, $q = O(r^{-n})$, w is bounded and $u - w \geq a$, it is easy to see that $f = O(r^{-n})$. Also

$$\Delta w - qw = kw - q_+w + q_-w > 0$$

because $k > q_+$, $w > 0$, where q_- is the negative part of q . We conclude that $f > 0$. On the other hand,

$$\begin{aligned}\Delta(u - w) - (q - f)(u - w) &= (\Delta u - qu) - \Delta w + qw + f(u - w) \\ &= 0.\end{aligned}$$

Hence the first Dirichlet eigenvalue of $\Delta - q + f$ is nonnegative by [FS, p.201].

Example: It is easy to construct examples of q such that $q = O(r^{-n})$ and the first eigenvalue of $L = \Delta - q$ is nonnegative, but the positive solution u of $Lu = 0$ has the property that $\lim_{x \rightarrow \infty} u = 0$. In fact, one can construct example with q satisfying the additional property that q has compact support and in particular $q \in L^1(M)$. First, let $q' \leq 0$ be a smooth function with compact support such that $q' < 0$ somewhere. Let u be a positive solution of $\Delta u = q'$ such that $u \rightarrow 0$ near infinity. Then u is a positive solution of $\Delta u - qu = 0$, where $q = q'/u$ which is smooth with compact support. Note that by the uniqueness result in Lemma 2.4, for this q every other positive solution of $\Delta v - qv = 0$ must be asymptotically zero.

Let M be an AF manifold and let $\Lambda > 0$ be the Sobolev constant on M . Namely,

$$(2.3) \quad \Lambda = \inf \left\{ \frac{\int_M |\nabla f|^2}{\left(\int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}} \mid f \in C_0^\infty(M), f \not\equiv 0 \right\}.$$

Since M is AF, $\Lambda > 0$, Theorem 2.1 implies the following result in [SY1].

Corollary 2.1. *Let (M^n, g) be an AF manifold and let Λ be the Sobolev constant for M . Let q be a smooth function on M such that $q = O(r^{-n})$. Suppose*

$$a = \left(\int_M q_-^{\frac{n}{2}} \right)^{\frac{2}{n}} < \Lambda.$$

where q_- is the negative part of q . Then $\Delta u - qu = 0$ has a positive solution u such that $u = 1 + O(r^{2-n})$.

Proof. For any smooth function f with compact support,

$$\begin{aligned}(2.4) \quad \int_M |\nabla f|^2 + \int_M q f^2 &= \int_M |\nabla f|^2 - \int_M q_- f^2 + \int_M q_+ f^2 \\ &\geq (\Lambda - a) \left(\int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}.\end{aligned}$$

Since $\Lambda - a > 0$, we can find a smooth function $h \geq 0$ with compact support such that $h \not\equiv 0$ and

$$\left(\int_M h^{\frac{n}{2}} \right)^{\frac{2}{n}} \leq \Lambda - a.$$

By (2.4), we have that

$$\int_M |\nabla f|^2 + \int_M q f^2 \geq \left(\int_M h^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \geq \int_M h f^2,$$

and hence the first Dirichlet eigenvalue of $\Delta - (q - h)$ is nonnegative. The corollary then follows from Theorem 2.1

Corollary 2.2. *Let (M^n, g) be an AF manifold with scalar curvature \mathcal{R} . Suppose the first Dirichlet eigenvalue of $\Delta - [(n-2)/4(n-1)\mathcal{R} - f]$ is nonnegative for some smooth function $f \geq 0$, $f \not\equiv 0$ and $f = O(r^{-n})$. In addition, suppose there is $\alpha > 0$ such that the Hölder norm of \mathcal{R} with exponent α in $B_x(\frac{1}{2}r_x)$ decays like $r_x^{-n-\alpha}$. Then M is conformally scalar flat in the sense that there is a smooth positive function u such that $(M, u^{\frac{4}{n-2}}g)$ is an AF manifold with zero scalar curvature.*

Proof. By Theorem 2.1 and the fact that $\mathcal{R} = O(r^{-n})$, there is a positive solution u of $\Delta u - \mathcal{R}u = 0$ such that $u(x) = 1 + O(r^{2-n})$ as $x \rightarrow \infty$. By the assumption on the Hölder norm of \mathcal{R} and the fact that g is AF, we have $|\nabla u| = O(r^{1-n})$ and $|\nabla^2 u| = O(r^{-n})$ by the Schauder estimate [GT, Theorem 6.2]. Hence $u^{\frac{4}{n-2}}g$ is also AF and has zero scalar curvature.

Remark 2.1. *Suppose the first Dirichlet eigenvalue of $\Delta - (n-2)/4(n-1)\mathcal{R}$ is nonnegative on M , then the first Dirichlet eigenvalue of this operator will be positive on every compact domain. Hence*

$$\int_M |\nabla f|^2 + \frac{n-2}{4(n-1)} \mathcal{R} f^2 > 0$$

for all $f \in C_0^\infty(M)$. From the example before Corollary 2.1, one can see that this may not imply that M is conformally scalar flat. Hence the result in [CB, Theorem 2.1, (I) implies (II)] seems to be incorrect. This has also been noticed by Maxwell [Ma]. In fact, Maxwell has obtained results similar to Corollary 2.2 with conditions in terms of positivity of certain Sobolev quotient, see [Proposition 4.1, Ma]. Moreover, AF manifolds with boundary are also discussed there and hence the results are more general than Corollary 2.2.

Using Theorem 2.1, one can find a harmonic spinor on M which is asymptotically parallel near infinity under certain condition on \mathcal{R} . Denote the Dirac operator on M by \mathcal{D} .

Theorem 2.2. *Let (M^n, g) be a spin AF manifold with scalar curvature \mathcal{R} . Suppose there is a smooth function $f \geq 0$, $f \not\equiv 0$ and $f = O(r^{-n})$ such that the first Dirichlet eigenvalue of $\Delta - (\frac{1}{4}\mathcal{R} - f)$ is nonnegative. Then for any spinor Ψ_0 in \mathbb{R}^n which is parallel with respect to the Euclidean metric, there exists a unique*

harmonic spinor Ψ on M such that $\mathcal{D}\Psi = 0$ and there is a constant C such that $|\Psi - \Psi_0| \leq Cr^{2-n}$ at infinity, where $|\cdot|$ is the norm with respect to g .

Proof. Let Ψ_0 be a parallel spinor over \mathbb{R}^n near infinity. Ψ_0 can be considered as a spinor over M near infinity. Extend Ψ_0 to be smooth on M . As in [ST1, §3], let p be a fixed point, for any $R > 0$ we first solve:

$$(2.5) \quad \begin{cases} \mathcal{D}^2 \sigma_R &= -\mathcal{D}^2 \Psi_0 \text{ in } B_p(R), \\ \sigma_R|_{\partial B_p(R)} &= 0 \end{cases}$$

To prove the solution exists, it is sufficient to prove that

$$(2.6) \quad a(\Psi, \Psi) = \int_{B(R)} \langle \mathcal{D}\Psi, \mathcal{D}\Psi \rangle \geq \delta \int_{B(R)} (|\nabla \Psi|^2 + |\Psi|^2)$$

for some $\delta > 0$ for all $\Psi \in W_0^{1,2}(B_p(R))$. By Lichnerowicz formula:

$$\begin{aligned} a(\Psi, \Psi) &= \int_{B_p(R)} \langle \nabla \Psi, \nabla \Psi \rangle + \frac{1}{4} \mathcal{R} \langle \Psi, \Psi \rangle \\ &= \int_{B_p(R)} |\nabla \Psi|^2 + \frac{1}{4} \mathcal{R} |\Psi|^2 \\ &= (1 - \tau) \left[\int_{B_p(R)} |\nabla \Psi|^2 + \frac{1}{4} \int_{B_p(R)} \mathcal{R} |\Psi|^2 \right] + \tau \left[\int_{B_p(R)} |\nabla \Psi|^2 + \frac{1}{4} \int_{B_p(R)} \mathcal{R} |\Psi|^2 \right] \\ &\geq (1 - \tau) \left[\int_{B_p(R)} |\nabla \Psi|^2 + \frac{1}{4} \int_{B_p(R)} \mathcal{R} |\Psi|^2 \right] + \lambda \tau \int_{B(R)} |\Psi|^2 \\ &\geq (1 - \tau) \int_{B(R)} |\nabla \Psi|^2 + \left(\frac{1 - \tau}{4} \inf_{B(R)} \mathcal{R} + \lambda \tau \right) \int_{B(R)} |\Psi|^2 \end{aligned}$$

where $0 < \tau < 1$ is a constant and $\lambda > 0$ is the first eigenvalue of $\Delta - \frac{1}{4}\mathcal{R}$ in $B_p(R)$ which is positive by assumption. Choose τ close enough to 1, (2.6) follows.

By Theorem 2.1, we can find a positive solution u of

$$(2.7) \quad \Delta u - \frac{1}{4} \mathcal{R} u = 0$$

such that $u \rightarrow 1$ at infinity. Without loss of generality, we assume that the norm of Ψ_0 is 1 with respect to the Euclidean metric near infinity. Let $\Psi_R = \sigma_R + \Psi_0$, then $\mathcal{D}^2 \Psi = 0$ on $B_p(R)$. We want to prove that Ψ_R is uniformly bounded. By Lichnerowicz formula, we have

$$\frac{1}{2} \Delta |\Psi_R|^2 \geq \frac{1}{4} \mathcal{R} |\Psi_R|^2 + |\nabla \Psi_R|^2$$

and so

$$\Delta|\Psi_R| \geq \frac{1}{4}\mathcal{R}|\Psi_R|.$$

Since $|\Psi_R|^2 = |\Psi_0|^2$ on $\partial B_p(R)$ and $|\Psi_0|$ is asymptotically 1 with respect to g as g is an AF metric, by Lemma 2.1, we conclude that given any $\epsilon > 0$,

$$|\Psi_R| \leq u + \epsilon$$

in $B_p(R)$ if R is large enough. Hence passing to a subsequence, Ψ_R converges to a spinor Ψ such that $\mathcal{D}^2\Psi = 0$ and

$$(2.8) \quad |\Psi| \leq u$$

on M .

To prove that Ψ is harmonic, by Lemma 3.4 in [ST1] it is sufficient to prove that

$$(2.9) \quad \int_M |\mathcal{D}\Psi|^2 < \infty.$$

For each R , $\Psi_R - \Psi_0 = \sigma_R = 0$ on $\partial B(R)$, so

$$\int_{B(R)} \langle \mathcal{D}\Psi_R, \mathcal{D}(\Psi_R - \Psi_0) \rangle = 0.$$

So

$$\int_{B(R)} |\mathcal{D}\Psi_R|^2 \leq \int_M |\mathcal{D}\Psi_0|^2.$$

Let $R \rightarrow \infty$, we see that (2.9) is true.

Next, we want to estimate $|\Psi - \Psi_0|$. By Lichnerowicz formula, for each R , we have

$$\Delta|\Psi_R - \Psi_0| \geq -|\mathcal{D}^2\Psi_0| + \frac{1}{4}\mathcal{R}|\Psi - \Psi_0|.$$

Since $|\mathcal{R}| = O(r^{-n})$ and $|\mathcal{D}^2\Psi_0| = O(r^{-n})$ and the fact that $|\Psi - \Psi_0|$ is bounded by (2.7), we have that

$$\Delta|\Psi_R - \Psi_0| \geq -C_1(1+r)^{-n}$$

for some positive constant C_1 . By Lemma 2.2, one can prove that $|\Psi - \Psi_0| \leq C_2 r^{2-n}$ near infinity for some constant C_2 .

Suppose Ψ_1 and Ψ_2 are two harmonic spinors such that $|\Psi_1 - \Psi_0| \rightarrow 0$ and $|\Psi_2 - \Psi_0| \rightarrow 0$ at infinity. By Lichnerowicz formula, we have

$$\Delta|\Psi_1 - \Psi_2| \geq \frac{1}{4}\mathcal{R}|\Psi_1 - \Psi_2|.$$

By Lemma 2.1, we can conclude that $|\Psi_1 - \Psi_2| \leq \epsilon u$ for any $\epsilon > 0$. Hence $\Psi_1 = \Psi_2$. This completes the proof of the theorem.

For simplicity, the harmonic spinor obtained in the theorem is said to be the harmonic spinor with boundary value Ψ_0 .

We can express the mass of M in terms of the harmonic spinor as in [W, PT]:

Corollary 2.3. *Let (M^n, g) be an AF manifold satisfying the conditions of Theorem 2.2. Let Ψ be the harmonic spinor on M with boundary value Ψ_0 where Ψ_0 is a parallel spinor with respect to the Euclidean metric such that $|\Psi_0| = 1$ near infinity. Then the mass of M is given by*

$$c(n)\mathfrak{m} = \int_M \left(|\nabla \Psi|^2 + \frac{1}{4}\mathcal{R}|\Psi|^2 \right).$$

where $c(n)$ is a positive constant depending only on n .

§3 Some lower bounds of ADM mass.

In this section, we shall give some lower bounds of the ADM mass of an AF manifold without assuming that the scalar curvature is nonnegative. Let (M^n, g) be an AF manifold with one end as in §2. In case $n \geq 4$, we assume that M is spin. Let $\Lambda > 0$ be the Sobolev constant on M defined in (2.3). Hence if $f \in C_0^\infty(M)$, then

$$(3.1) \quad \int_M |\nabla f|^2 \geq \Lambda \left(\int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n}{n-2}}.$$

It is easy to see that (3.2) is still true if f is smooth such that $|f| = O(r^{-\tau})$ with $\tau > (n-2)/2$.

Theorem 3.1. *Let (M^n, g) be a spin AF manifold with scalar curvature \mathcal{R} . Let \mathcal{R}_+ and \mathcal{R}_- be the positive and negative part of \mathcal{R} respectively. Suppose*

$$(3.2) \quad a = \left(\int_M \left(\frac{\mathcal{R}_-}{4} \right)^{\frac{n}{2}} \right)^{\frac{2}{n}} < \frac{\Lambda}{2}.$$

Then the mass \mathfrak{m} of M has a lower bound given by

$$\mathfrak{m} \geq C(n) \frac{\Lambda - 2a}{\Lambda + b - a} \left(\int_M \mathcal{R}_+ - \frac{\Lambda + 2b}{\Lambda - 2a} \int_M \mathcal{R}_- \right),$$

where $C(n)$ is a positive constant depending only on n and

$$b = \left(\int_M \left(\frac{\mathcal{R}_+}{4} \right)^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

In particular, if $\mathcal{R} \geq 0$ then

$$\mathfrak{m} \geq C(n) \frac{\Lambda}{\Lambda + b} \int_M \mathcal{R}.$$

Proof. By Corollary 2.1, there is a positive solution u of

$$(3.3) \quad \Delta u - \frac{\mathcal{R}}{4} u = 0$$

with $u = 1 + (r^{2-n})$. By Theorem 2.2, there is a harmonic spinor Ψ such that its norm is asymptotically equal to 1 near infinity. Namely, $|\Psi| = 1 + O(r^{2-n})$. Let $v = |\Psi|$, then the ADM mass of M satisfies

$$(3.4) \quad \begin{aligned} C(n)\mathfrak{m} &= \int_M \left(|\nabla \Psi|^2 + \frac{\mathcal{R}}{4} |\Psi|^2 \right) \\ &\geq \int_M \left(|\nabla v|^2 + \frac{\mathcal{R}}{4} v^2 \right) \\ &\geq \int_M |\nabla(v-1)|^2 + \left(1 - \frac{1}{\epsilon}\right) \int_M \frac{\mathcal{R}_+}{4} (v-1)^2 - \left(1 + \frac{1}{\epsilon}\right) \int_M \frac{\mathcal{R}_-}{4} (v-1)^2 \\ &\quad + (1-\epsilon) \int_M \frac{\mathcal{R}_+}{4} - (1+\epsilon) \int_M \frac{\mathcal{R}_-}{4} \\ &\geq \left(\Lambda - \left(\frac{1}{\epsilon} - 1 \right) b - \left(1 + \frac{1}{\epsilon} \right) a \right) \left(\int_M (v-1)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + (1-\epsilon) \int_M \frac{\mathcal{R}_+}{4} \\ &\quad - (1+\epsilon) \int_M \frac{\mathcal{R}_-}{4}. \end{aligned}$$

for any $0 < \epsilon \leq 1$, where we have used the fact that $|v-1| = O(r^{2-n})$ and the Sobolev inequality (3.1). Choose ϵ such that

$$\Lambda - \left(\frac{1}{\epsilon} - 1 \right) b - \left(1 + \frac{1}{\epsilon} \right) a = 0.$$

That is to say

$$\epsilon = \frac{b+a}{\Lambda+b-a}.$$

Note that $0 < \epsilon \leq 1$ unless $\mathcal{R} \equiv 0$ because $\Lambda > 2a$. If $\mathcal{R} \equiv 0$, then the theorem is true by the positive mass theorem [SY1-2, W, PT]. If $\mathcal{R} \not\equiv 0$, then the theorem follows from (3.4) and the definition of ϵ .

Corollary 3.1. *Same assumptions and notations as in Theorem 3.1. Suppose*

$$\int_M \mathcal{R}_+ \geq \frac{\Lambda + 2b}{\Lambda - 2a} \int_M \mathcal{R}_-$$

then the mass \mathfrak{m} of M is nonnegative. Moreover, $\mathfrak{m} = 0$ if and only if M is the Euclidean space.

Proof. The fact that $\mathfrak{m} \geq 0$ follows immediately from the theorem. Suppose $\mathfrak{m} = 0$, then every inequality in (3.4) becomes an inequality. Hence \mathcal{R}_+ , $(v-1)^2$ and \mathcal{R}_- are proportional to each other. So $\mathcal{R} \equiv 0$ and M is the Euclidean space by the standard positive mass theorem.

Remark 3.1. *The result of Corollary 3.1 can also be derived from the result in [ZZ] under similar conditions. More precisely, if we replace $\mathcal{R}_+/4$ and $\mathcal{R}_-/4$ by $\mathcal{R}_+/8$ and $\mathcal{R}_-/8$ in the assumptions of the corollary, then the proof of the result in [ZZ, Theorem 4.1] together with similar derivation as in (3.4), we may also conclude that $\mathfrak{m} \geq 0$.*

Under the assumptions of Theorem 3.1 or the assumptions in [ZZ, Theorem 4.1], we must have $\int_M \mathcal{R} \geq 0$. One might ask what might happen if $\int_M \mathcal{R} < 0$. In this situation, we want to give a lower bound for the ADM mass using the methods in [BF, FK] and we shall give another condition so that the ADM mass is nonnegative. In the following, we assume that (M^n, g) is an AF manifold with scalar curvature \mathcal{R} which may be negative somewhere. We always assume that the operator $\Delta - 1/4\mathcal{R} + f$ has nonnegative Dirichlet eigenvalue for some smooth function $f \geq 0$, $f \not\equiv 0$ and $f = O(r^{-n})$. Hence

$$(3.5) \quad \Delta u - \frac{1}{4}\mathcal{R}u = 0$$

has a positive solution which tends to 1 near infinity by Theorem 2.1. Denote

$$(3.6) \quad A = \sup_M u.$$

We will follow the arguments in [FK]. Let Ψ_0 be a spinor which is parallel near infinity with respect to the Euclidean metric such that $|\Psi_0|_e = 1$ near infinity, where $|\cdot|_e$ is the norm of Ψ_0 with respect to the Euclidean metric. By Theorem 2.2, because of (3.5), we can find a harmonic spinor Ψ on M such that $|\Psi - \Psi_0| \rightarrow 0$ and the ADM mass \mathfrak{m} is given by

$$C(n)\mathfrak{m} = \int_M \left(|\nabla \Psi|^2 + \frac{1}{4}\mathcal{R}|\Psi|^2 \right).$$

where $C(n) > 0$ is a constant depending only on n . For such a Ψ , define

$$(3.7) \quad \mathfrak{m}_\Psi = C(n)\mathfrak{m} - \frac{1}{4} \int_M \mathcal{R}|\Psi|^2 = \int_M |\nabla \Psi|^2.$$

Hence $\mathfrak{m}_\Psi \geq 0$ for all such Ψ . Let Ψ_0 and Ψ as above. We have the following:

Lemma 3.1. *For all $x \in M$,*

$$(3.7) \quad |\Psi_x| \leq u(x) \leq A$$

where u is the positive solution in (3.5) and A is given by (3.6).

Proof. By the Lichnerowicz formula,

$$\frac{1}{2}\Delta|\Psi|^2 = |\nabla\Psi|^2 + \frac{1}{4}\mathcal{R}|\Psi|^2$$

and so

$$\Delta|\Psi| \geq \frac{1}{4}\mathcal{R}|\Psi|.$$

Since $|\Psi| \rightarrow 1$ near infinity, the lemma follows by maximum principle Lemma 2.2.

Let Ψ be as in Lemma 3.1. We have:

Lemma 3.2. *With the above notations, let $\eta \geq 0$ be a smooth function on M such that $\sup_M(|\eta| + |\Delta\eta|) < \infty$, we have*

$$(3.8) \quad \int_M \eta \|\nabla^2 \Psi\|^2 \leq C_1 \mathfrak{m}_\Psi \sup_M (|\eta Rm| + |\Delta\eta|) + C_2 A \sqrt{\mathfrak{m}_\Psi} \|\eta \nabla Rm\|_2$$

for some constants C_1, C_2 depending only on n

Proof. The proof is exactly as in Corollary 3.2 in [FK], except in the last part, we have to use Lemma 3.1 and the definition of \mathfrak{m}_Ψ .

Let $N = 2^{\lfloor \frac{n}{2} \rfloor}$. Choose an orthonormal basis of constant spinors Ψ_0^i , $1 \leq i \leq N$ with respect to the Euclidean metric and let Ψ^i be the corresponding harmonic spinors. For $x \in M$, define P_x as in [FK]. Namely,

$$P_x : S_x(M) \rightarrow S_x(M)$$

where $S_x(M)$ is the fibre of the spinor bundle associated with the spin structure through the spinor representation and

$$(3.9) \quad P_x(\Psi) = \sum_{i=1}^n \langle \Psi_x^i, \Psi_x \rangle \Psi_x^i.$$

Note that P_x will tend to the identity map near infinity.

Lemma 3.3. *$|P_x| \leq A$, where $|P_x|$ is the norm of the operator P_x and A is given by (3.6).*

Proof. The proof is same as Lemma 4.1 in [FK], except we have to use Lemma 3.1.

Lemma 3.4. *There is a constant c depending only on n such that for any $\epsilon > 0$*

$$\|Id - P_x\|^2 < \epsilon$$

except on a set $D(\epsilon)$ with

$$\text{vol}(D(\epsilon)) \leq \left(\frac{c \sum_{i=1}^N \mathfrak{m}_{\Psi_i}}{\epsilon^2 \Lambda} \right)^{\frac{n}{n-2}}$$

where Id is the identify map and Λ is the Sobolev constant given in (3.1).

Proof. The proof is exactly as in Lemma 4.2 in [FK].

Now choose $\epsilon = N/32$, then outside $D = D(\epsilon)$, $\|Id - P\| < \sqrt{N/32}$, and we have

$$\frac{N}{2} |Rm|^2 \leq 32 \sum_{i=1}^N \|\nabla^2 \Psi^i\|^2$$

by Lemma 5.1 in [FK]. Combining this with (3.8), we have

$$\begin{aligned} \int_{M \setminus D} \eta |Rm|^2 &\leq C(n) \int_{M \setminus D} \sum_{i=1}^N \|\nabla^2 \Psi^i\|^2 \\ &\leq C_1 \left(\sum_{i=1}^N \mathfrak{m}_{\Psi_i} \right) \sup_M (|\eta Rm| + |\Delta \eta|) + C_2 A \sum_{i=1}^N \sqrt{\mathfrak{m}_{\Psi_i}} \|\eta \nabla Rm\|_2 \end{aligned}$$

where C_1 and C_2 are constants depending only on n . Hence we have the following:

Theorem 3.2. *Let Ψ^i , $1 \leq i \leq N$ as above. Then there are constants $C_1(n)$, $C_2(n)$ depending only on n such that for any smooth function η on M with $\sup_M (|\eta| + |\Delta \eta|) < \infty$, we have that*

$$(3.10) \quad \int_{M \setminus D} \eta |Rm|^2 \leq C_1 \left(\sum_{i=1}^N \mathfrak{m}_{\Psi_i} \right) \sup_M (|\eta Rm| + |\Delta \eta|) + C_2 A \sum_{i=1}^N \sqrt{\mathfrak{m}_{\Psi_i}} \|\eta \nabla Rm\|_2.$$

where D is a set with

$$\text{vol}(D) \leq \left(\frac{c \sum_{i=1}^N \mathfrak{m}_{\Psi_i}}{\left(\frac{N}{32}\right)^2 \Lambda} \right)^{\frac{n}{n-2}}$$

where c is the constant in Lemma 3.4.

Let \mathcal{R}_- is the negative part of \mathcal{R} and let

$$(3.11) \quad B = \inf_D \left\{ \int_{M \setminus D} |Rm|^2 \mid \text{vol}(D) \leq \left[\frac{cAN \int_M \mathcal{R}_-}{\left(\frac{N}{32}\right)^2 \Lambda} \right]^{\frac{n}{n-2}} \right\}.$$

Corollary 3.2. *Let (M^n, g) be a spin AF manifold with scalar curvature \mathcal{R} such that the operator $\Delta - \frac{1}{4}\mathcal{R} + f$ has nonnegative Dirichlet eigenvalue for some smooth function $f \geq 0$, $f \not\equiv 0$, $f = O(r^{-n})$. There exists $C(n) > 0$ depending only on n such that if*

$$B \geq C(n)A^2 \left[\sup_M |Rm| \int_M R_- + \|\nabla Rm\|_2 \left(\int_M R_- \right)^{\frac{1}{2}} \right]$$

then the mass of M is nonnegative, where A is given by (3.6) and B is given by (3.11).

Proof. If M is flat, then it is obvious that $\mathfrak{m} = 0$. Suppose M is non flat and suppose $\mathfrak{m} < 0$, then $\mathcal{R} < 0$ somewhere by the positive mass theorem. From the definitions of \mathfrak{m}_{Ψ^i} and A and by Lemma 3.1, we have that

$$(3.12) \quad \mathfrak{m}_{\Psi^i} < \frac{A^2}{4} \int_M R_-$$

for all $1 \leq i \leq N$. Take $\eta \equiv 1$ in (3.10), we have that

$$\begin{aligned} B &\leq \int_{M \setminus D} |Rm|^2 \\ &\leq C_1 \left[\left(\sum_{i=1}^N \mathfrak{m}_{\Psi^i} \right) \sup_M |Rm| + A \left(\sum_{i=1}^N \sqrt{\mathfrak{m}_{\Psi^i}} \right) \|\nabla Rm\|_2 \right] \\ &< C_2 A^2 \left[\sup_M |Rm| \int_M R_- + \|\nabla Rm\|_2 \left(\int_M R_- \right)^{\frac{1}{2}} \right]. \end{aligned}$$

for some positive constants C_1, C_2 depending only on n because M is nonflat. From this, the result follows.

Under certain conditions, we can estimate A from above. For example, we have the following:

Corollary 3.3. *Let (M^3, g) be an AF manifold with scalar curvature \mathcal{R} such that*

$$\left(\int_M \left(\frac{\mathcal{R}_-}{4} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} < \Lambda,$$

where Λ is the Sobolev constant of M . Then there is a constant $C > 0$ depending only on n and positive constant A depending only on \mathcal{R}_- and Λ such that if

$$B \geq CA^2 \left[\sup_M |Rm| \int_M R_- + \|\nabla Rm\|_2 \left(\int_M R_- \right)^{\frac{1}{2}} \right]$$

where B is as in Corollary 3.2. Then $\mathfrak{m} \geq 0$.

Proof. By Corollary 2.1, M satisfies the conditions in Corollary 3.2. It remains to prove that A in Corollary 3.2 is less than some constant C depending only on \mathcal{R}_- and Λ . This will be proved in Corollary 4.1, next section.

§4 Nonnegativity and lower estimates of Brown-York mass.

Suppose (Ω^3, g) is a 3-dimensional compact manifold with smooth boundary. For simplicity, in this section we always assume that $\partial\Omega = \Sigma$ is connected. Suppose the Gauss curvature of Σ is positive, then by a classical result we know that Σ can be isometrically embedding into \mathbb{R}^3 , see [N]. Let H_0 be the mean curvature of the embedding image in \mathbb{R}^3 with respect to the outward unit norm. Then the Brown-York mass of (Ω, g) is defined as follow (see [BY]):

$$(4.1) \quad \mathfrak{m}_{\text{BY}}(\Omega) = \int_{\Sigma} (H_0 - H) d\sigma,$$

where $d\sigma$ is the volume element of Σ , H is the mean curvature of $\partial\Omega$ with respect to original metric g and outward unit normal.

In [ST1], it was proved that if (i) the scalar curvature of (Ω, g) is nonnegative and (ii) $H > 0$, then $\mathfrak{m}_{\text{BY}}(\Omega) \geq 0$. Moreover, equality holds if and only if Ω is a domain in \mathbb{R}^3 . In this section, we will discuss cases that either (i) or (ii) does not hold. As for ADM mass of an AF manifold, we will also give some lower estimates of the Brown-York mass of a bounded domain.

In [ST2], some results in [ST1] are generalized to the case that the Gauss curvature of Σ is only assumed to be nonnegative. We believe that some of the results in this section are still true if Σ is only assumed to have nonnegative Gauss curvature. However, we always assume that the Gauss curvature of Σ is positive for simplicity.

Let us first consider the case that the scalar curvature \mathcal{R} may be negative somewhere. The idea is to solve

$$(4.2) \quad \begin{cases} \Delta u - qu &= 0 \text{ in } \Omega \\ u &= 1 \text{ on } \partial\Omega \end{cases}$$

If (4.2) has a positive solution with $q = \frac{\mathcal{R}}{8}$, then the metric $g_1 = u^4 g$ has zero scalar curvature. We can then apply the result of [ST1] provided that the mean curvature of Σ with respect to g_1 is positive. This will be true if $\frac{\partial u}{\partial \nu}$ is not too negative, where ν is the unit outward normal of Σ . To this end, we first get some upper estimates of the solution u of (4.2). We have the following:

Lemma 4.1. *Let (Ω^3, g) be a compact manifold with smooth boundary and let q be a smooth function defined on $\overline{\Omega}$. Suppose*

$$(4.3) \quad \beta < \Lambda$$

then (4.2) has a unique solution u and such that

$$(4.4) \quad 0 < u \leq 1 + 27^{\frac{1}{8}} \gamma \left[\frac{(\alpha + 1)(1 + \Lambda - \beta)}{\Lambda(\Lambda - \beta)} + 1 \right]$$

in Ω where $\alpha = \max_{\Omega} q_-$, $\beta = \left(\int_{\Omega} q_-^{\frac{3}{2}} \right)^{\frac{2}{3}}$, $\gamma = \sup_{p \geq 1} \left(\int_{\Omega} q_-^p \right)^{\frac{1}{p}}$, q_- is the negative part of q and Λ is the Sobolev constant of Ω .

Proof. Let $f \in C_0^\infty(\Omega)$ and $f \not\equiv 0$, then

$$\begin{aligned} \int_{\Omega} (|\nabla f|^2 + q f^2) &\geq \int_{\Omega} (|\nabla f|^2 - q_- f^2) \\ &\geq (\Lambda - \beta) \left(\int_{\Omega} f^6 \right)^{\frac{1}{3}} \\ &\geq (\Lambda - \beta) |\Omega|^{-\frac{2}{3}} \int_{\Omega} f^2. \end{aligned}$$

Since $\Lambda - \beta > 0$, (4.2) has a unique positive solution u by [GT, Theorem?]. Let $v = u - 1$, then

$$(4.5) \quad \begin{cases} \Delta v - qv &= q \text{ in } \Omega \\ v &= 0 \text{ on } \partial\Omega. \end{cases}$$

Let $v_+ = \max\{v, 0\}$. For any integer $k \geq 1$, multiply the first equation in (4.5) by v_+^{2k-1} and integrating by parts, we have that

$$\begin{aligned} \int_{\Omega} q_- (v_+^{2k-1} + v_+^{2k}) &\geq - \int_{\Omega} q (v_+^{2k-1} + v_+^{2k}) \\ &= (2k - 1) \int_{\Omega} v_+^{2k-2} |\nabla v_+|^2 \\ &= \frac{2k - 1}{k^2} \int_{\Omega} |\nabla v_+^k|^2 \\ &\geq \frac{\Lambda}{k} \left(\int_{\Omega} v_+^{6k} \right)^{\frac{1}{3}} \end{aligned}$$

where we have used the fact that $v = 0$ on $\partial\Omega$. Hence

$$\begin{aligned} (4.6) \quad \left(\int_{\Omega} v_+^{6k} \right)^{\frac{1}{3}} &\leq \frac{k}{\Lambda} \left[\alpha \int_{\Omega} v_+^{2k} + \left(\int_{\Omega} v_+^{2k} \right)^{\frac{2k-1}{2k}} \left(\int_{\Omega} q_-^{2k} \right)^{\frac{1}{2k}} \right] \\ &\leq \frac{k}{\Lambda} \left[\left(\frac{2k-1}{2k} + \alpha \right) \int_{\Omega} v_+^{2k} + \frac{1}{2k} \int_{\Omega} q_-^{2k} \right] \\ &\leq \frac{(\alpha + 1)k}{\Lambda} \max \left\{ \int_{\Omega} v_+^{2k}, \int_{\Omega} q_-^{2k} \right\} \end{aligned}$$

where we have used Hölder and Young inequalities. For any $k \geq 1$, let

$$I_k = \left(\int_{\Omega} v_+^{2k} \right)^{\frac{1}{2k}}$$

and let $a = (\alpha + 1)/\Lambda$. By (4.6) and the definition of γ , we have that

$$(4.7) \quad \begin{aligned} I_{3k} &\leq \left(\frac{(\alpha + 1)k}{\Lambda} \right)^{\frac{1}{2k}} \max\{I_k, \gamma\} \\ &= (ak)^{\frac{1}{2k}} \max\{I_k, \gamma\}. \end{aligned}$$

Let $\ell_0 \geq 0$ be such that $ak_0 = a3^{\ell_0} \geq 1$. We claim that for $\ell \geq 1$,

$$(4.8) \quad \begin{aligned} I_{3^\ell k_0} &\leq a^{\frac{1}{2k_0} \sum_{i=0}^{\ell-1} 3^{-i}} k_0^{\frac{1}{2k_0} \sum_{i=0}^{\ell-1} 3^{-i}} 3^{\frac{1}{2k_0} \sum_{i=0}^{\ell-1} i 3^{-i}} \max\{I_{k_0}, \gamma\} \\ &= a^{\frac{1}{2} \sum_{i=\ell_0}^{\ell_0+\ell-1} 3^{-i}} 3^{\frac{1}{2} \sum_{i=\ell_0}^{\ell_0+\ell-1} i 3^{-i}} \max\{I_{k_0}, \gamma\}. \end{aligned}$$

If $\ell = 1$, then (4.8) is true by (4.7). Suppose (4.8) is true for ℓ , by (4.7), we have that

$$(4.9) \quad \begin{aligned} &I_{3^{\ell+1}k_0} \\ &\leq (a \cdot 3^\ell k_0)^{\frac{1}{2 \cdot 3^\ell k_0}} \max\{I_{3^\ell k_0}, \gamma\} \\ &\leq (a \cdot 3^\ell k_0)^{\frac{1}{2 \cdot 3^\ell k_0}} \max \left\{ a^{\frac{1}{2k_0} \sum_{i=0}^{\ell-1} 3^{-i}} k_0^{\frac{1}{2k_0} \sum_{i=0}^{\ell-1} 3^{-i}} 3^{\frac{1}{2k_0} \sum_{i=0}^{\ell-1} i 3^{-i}} \max\{I_{k_0}, \gamma\}, \gamma \right\} \\ &\leq a^{\frac{1}{2k_0} \sum_{i=0}^{\ell} 3^{-i}} k_0^{\frac{1}{2k_0} \sum_{i=0}^{\ell} 3^{-i}} 3^{\frac{1}{2k_0} \sum_{i=0}^{\ell} i 3^{-i}} \max\{I_{k_0}, \gamma\} \end{aligned}$$

where we have used the fact that $ak_0 \geq 1$. Hence (4.8) is true. Suppose ℓ_0 satisfies $a3^\ell < 1$ for all $0 \leq \ell < \ell_0$, then for $k_0 = 3^{\ell_0}$, (4.7) implies

$$(4.10) \quad \begin{aligned} I_{k_0} &= I_{3^{\ell_0}} \\ &\leq (a \cdot 3^{\ell_0-1})^{\frac{1}{2 \cdot 3^{\ell_0-1}}} \max\{I_{3^{\ell_0-1}}, \gamma\} \\ &\leq \max \left\{ (a \cdot 3^{\ell_0-1})^{\frac{1}{2 \cdot 3^{\ell_0-1}}} I_{3^{\ell_0-1}}, \gamma \right\} \\ &\leq \dots \dots \\ &\leq \max \left\{ a^{\frac{1}{2} \sum_{i=0}^{\ell_0-1} 3^{-i}} 3^{\frac{1}{2} \sum_{i=1}^{\ell_0-1} i 3^{-i}} I_3, \gamma \right\} \end{aligned}$$

where we have used the fact that $a \cdot 3^\ell \leq 1$ if $\ell < \ell_0$. Note that

$$(4.11) \quad \lim_{\ell \rightarrow \infty} I_{3^\ell} = \lim_{\ell \rightarrow \infty} \left(\int_{\Omega} v_+^{3^\ell} \right)^{\frac{1}{3^\ell}} = \lim_{\ell \rightarrow \infty} \left(\frac{1}{|\Omega|} \int_{\Omega} v_+^{3^\ell} \right)^{\frac{1}{3^\ell}} = \sup_{\Omega} v^+.$$

Suppose $a \geq 1$, we take $\ell_0 = 1$ in (4.8), we have

$$I_{3^\ell} \leq a^{\frac{1}{2} \sum_{i=1}^{\ell} 3^{-i}} 3^{\frac{1}{2} \sum_{i=1}^{\ell} i 3^{-i}} \max\{I_3, \gamma\}.$$

Let $\ell \rightarrow \infty$, we have

$$(4.12) \quad \sup_{\Omega} v_+ \leq 27^{\frac{1}{8}} a \max\{I_3, \gamma\} \leq 27^{\frac{1}{8}} (a I_3 + (a+1)\gamma)$$

because $3^{\sum_{i=1}^{\infty} i 3^{-i}} = 27^{\frac{1}{4}}$. Suppose $a < 1$. Then choose $\ell_0 \geq 0$ such that $a k_0 = a 3^{\ell_0} \geq 1$ for $\ell \geq \ell_0$ and such that $a 3^\ell < 1$ for $0 \leq \ell < \ell_0$. By (4.8) and (4.10), we see that (4.12) is still true.

To estimate I_3 , as before we have

$$\begin{aligned} \Lambda \left(\int_{\Omega} v_+^6 \right)^{\frac{1}{3}} &\leq \int_{\Omega} q_- (v_+^2 + v_+) \\ &\leq \left(\int_{\Omega} q_-^{\frac{3}{2}} \right)^{\frac{2}{3}} \left(\int_{\Omega} v_+^6 \right)^{\frac{1}{3}} + \left(\int_{\Omega} q_-^{\frac{6}{5}} \right)^{\frac{5}{6}} \left(\int_{\Omega} v_+^6 \right)^{\frac{1}{6}} \end{aligned}$$

By the definitions of β and γ , we have

$$(\Lambda - \beta) I_3 \leq \gamma.$$

Combining this with (4.12), we have

$$\sup_{\Omega} v_+ \leq 27^{\frac{1}{8}} \gamma \left[\frac{(\alpha+1)(1+\Lambda-\beta)}{\Lambda(\Lambda-\beta)} + 1 \right].$$

Corollary 4.1. *Let (M^3, g) be an AF manifold with Sobolev constant Λ . Let q be a smooth function such that $q = O(r^{-3})$, and suppose*

$$\beta = \left(\int_M q_-^{\frac{3}{2}} \right)^{\frac{2}{3}} < \Lambda$$

Then $\Delta - q$ has a positive solution u which is asymptotically 1 near infinity such that

$$0 < u \leq 1 + 27^{\frac{1}{8}} \gamma \left[\frac{(\alpha+1)(1+\Lambda-\beta)}{\Lambda(\Lambda-\beta)} + 1 \right]$$

where $\alpha = \sup_M q_-$ and $\gamma = \sup_{p \geq 1} \left(\int_M q_-^p \right)^{\frac{1}{p}}$. In particular, u is bounded from above by a constant depending only on q_- and Λ .

Proof. The existence of u which is asymptotically 1 near infinity is a consequence of Corollary 2.1. It is easy to see that u is the limit of a sequence of solutions of

$$\begin{cases} \Delta u_k - q u_k &= 0 \text{ in } \Omega_k \\ u_k &= 1 \text{ on } \partial \Omega_k \end{cases}$$

where $\{\Omega_k\}_{k=1}^\infty$ is some family of bounded domains with smooth boundaries which exhausts M . The estimate of u follows from the lemma.

Lemma 4.2. *Let (Ω^3, g) be a compact manifold with smooth boundary Σ and with scalar curvature \mathcal{R} . Let*

$$I = \inf \left\{ \int_{\Omega} \left(|\nabla w|^2 + \frac{\mathcal{R}}{8} w^2 \right) \middle| w \text{ is smooth in } \overline{\Omega}, w \equiv 1 \text{ on } \partial \Omega \right\}.$$

- (i) Suppose $\beta = \left(\int_{\Omega} q_-^{\frac{3}{2}} \right)^{\frac{2}{3}} < \frac{\Lambda}{2}$, where Λ is the Sobolev constant of Ω , $q = \mathcal{R}/8$.

Then

$$I \geq \frac{\Lambda - 2\beta}{8(\Lambda + \delta - \beta)} \left(\int_{\Omega} \mathcal{R}_+ - \frac{\Lambda + 2\delta}{\Lambda - 2\beta} \int_{\Omega} \mathcal{R}_- \right)$$

$$\text{where } \delta = \left(\int_{\Omega} q_+^{\frac{3}{2}} \right)^{\frac{2}{3}}.$$

- (ii) Let λ be the first Dirichlet eigenvalue for the Laplacian of Ω . Suppose $8\lambda + \mathcal{R} > 0$ in Ω . Then

$$I \geq \lambda \int_{\Omega} \frac{\mathcal{R}}{8\lambda + \mathcal{R}}.$$

Proof. The proof of part (i) is similar to the proof of Theorem 3.1.

To prove (ii), let w be smooth in $\overline{\Omega}$, $w \equiv 1$ on $\partial \Omega$, and let $v = w - 1$, then

$$\begin{aligned} \int_{\Omega} \left(|\nabla w|^2 + \frac{\mathcal{R}}{8} w^2 \right) &= \int_{\Omega} \left(|\nabla v|^2 + \frac{\mathcal{R}}{8} (1+v)^2 \right) \\ &\geq \frac{1}{8} \int_{\Omega} (8\lambda v^2 + \mathcal{R}(1+v)^2) \\ &\geq \int_{\Omega} \frac{\lambda \mathcal{R}}{8\lambda + \mathcal{R}}. \end{aligned}$$

by minimizing $f(v) = 8\lambda v^2 + \mathcal{R}(1+v)^2$, where we have used the fact that $8\lambda + \mathcal{R} > 0$ in Ω . From this the lemma follows.

Now consider a compact manifold (Ω, g) with smooth boundary Σ with positive mean curvature with respect to the outward normal. Let $s_0 > 0$ be such that $d(x, \partial \Omega)$ is smooth in $\Omega_{s_0} = \{x \mid 0 < d(x, \partial \Omega) \leq s_0\}$ and such that the mean curvature of $\{x \mid d(x, \partial \Omega) = s\}$ with respect to the outward norm is positive for all $0 \leq s \leq s_0$. We have the following theorem which implies that if \mathcal{R} is not too negative, then $\mathfrak{m}_{\text{BY}}(\Omega)$ is still nonnegative.

Theorem 4.1. *Let (Ω^3, g) be a compact manifold with boundary with scalar curvature \mathcal{R} . With the above notations and assumptions, let H_{\min} be the minimum of the mean curvature of Σ with respect to the outward normal which is assumed to be positive and let*

$$(4.13) \quad \xi = \min\left\{\frac{\pi}{6s_0}, \frac{1}{2}H_{\min}\right\}.$$

Suppose the Gauss curvature of Σ is positive and suppose the following are true.

- (i) $\beta = \left(\int_{\Omega} q_-^{\frac{3}{2}}\right)^{\frac{2}{3}} < \Lambda$, where Λ is the Sobolev constant of Ω and $q = \mathcal{R}/8$.
- (ii) $\alpha = \max_{\Omega} q_- \leq \xi^2$.
- (iii) $\gamma = \sup_{p \geq 1} \left(\int_{\Omega} q_-^p\right)^{\frac{1}{p}} \leq 27^{-\frac{1}{8}} \left[\frac{(\alpha+1)(1+\Lambda-\beta)}{\Lambda-\beta} + 1\right]^{-1} \cdot \frac{\xi s_0}{10}$.

Then the Brown-York mass of Ω satisfies:

$$m_{\text{BY}}(\Omega) \geq \frac{1}{4} \inf \left\{ \int_{\Omega} \left(|\nabla w|^2 + \frac{\mathcal{R}}{8} w^2 \right) \mid w \equiv 1 \text{ on } \Sigma \right\}.$$

Remark 4.1. (a) *The conditions of the theorem are obviously satisfied if Ω has nonnegative scalar curvature.* (b) *The conditions (i)–(iii) of the theorem will be satisfied if $\alpha \leq \xi^2$, $\alpha|\Omega|^{\frac{2}{3}} \leq \Lambda/2$ and $\alpha \max\{|\Omega|, 1\} \leq 27^{-\frac{1}{8}} \left[\frac{(\xi^2+1)(2+\Lambda)}{\Lambda^2} + 1\right]^{-1} \cdot \frac{\xi s_0}{10}$. Hence, there is a constant C depending only on Λ , s_0 , H_{\min} and $|\Omega|$ such that if $\alpha \leq C$ then the conditions of the theorem will be satisfied.*

Proof of Theorem 4.1. By (i) and Lemma 4.1, there is a unique positive solution u of (4.2) with $q = \mathcal{R}/8$. Let $\phi(s) = \cos \xi s + \sin \xi s$, $0 \leq s \leq s_0$. On $0 \leq s \leq s_0$, $\phi > 0$ by (4.13),

$$(4.14) \quad \frac{d^2 \phi}{ds^2} = -\xi^2 \phi,$$

and

$$(4.15) \quad \begin{aligned} \frac{d\phi}{ds} &= \xi (-\sin \xi s + \cos \xi s) \\ &\geq \xi \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} \right) \\ &\geq \frac{\xi}{10}. \end{aligned}$$

Moreover,

$$(4.16) \quad \phi(0) = 1, \quad \phi(s_0) \geq 1 + \frac{\xi s_0}{10}, \quad \phi'(0) = \xi$$

Define $f(x) = \phi(d(x, \partial\Omega))$ for $x \in \Omega_{s_0}$. Then in Ω_{s_0}

$$(4.17) \quad \begin{aligned} \Delta f - qf &= \phi'' - H\phi' - q\phi \\ &\leq (-\xi^2 + q_-)\phi \\ &\leq 0 \end{aligned}$$

where H is the mean curvature of $\{d(x, \partial\Omega) = s\}$ and we have used the facts that $H > 0$, $\phi' \geq 0$, $\phi \geq 0$, and $a = \max_{\Omega} q_- \leq \xi^2$. Moreover, $f = 1$ on $\partial\Omega$, and on $\{d(x, \partial\Omega) = s_0\}$,

$$f \geq 1 + \frac{\xi s_0}{10} \geq u$$

by (4.15), condition (iii) and Lemma 4.1. Since the first Dirichlet eigenvalue of $\Delta - q$ in Ω_{s_0} is positive, by the maximum principle, we have $f \geq u$ in Ω_{s_0} . Hence on $\partial\Omega$

$$\frac{\partial u}{\partial \nu} \geq \frac{\partial f}{\partial \nu} = -\xi$$

where ν is the unit outward normal of Σ . Consider the metric $g_1 = u^4 g$, then the scalar curvature of g_1 is zero and the mean curvature \overline{H} with respect to g_1 of $\partial\Omega$ is

$$\overline{H} = H + \frac{\partial u}{\partial \nu} \geq H - \xi > 0$$

where H is the mean curvature with respect to g and we have used (4.13). Since $u = 1$ on the boundary, the induced metric on Σ is the same as before and so the Gauss curvature of Σ is positive. By [ST1], we have

$$\int_{\Sigma} (H_0 - \overline{H}) d\sigma \geq 0.$$

Since $\overline{H} = H + \frac{1}{4} \frac{\partial u}{\partial \nu}$

$$\mathfrak{m}_{\text{BY}}(\Omega) = \int_{\Sigma} (H_0 - H) d\sigma \geq \frac{1}{4} \int_{\Sigma} \frac{\partial u}{\partial \nu} d\sigma = \frac{1}{4} \int_{\Omega} |\nabla u|^2 + qu^2.$$

From this the result follows.

We should remark that in Lemma 4.1 without assuming $\beta < \Lambda$, we may obtained an upper bound for u in terms of Λ , α , $|\Omega|$ and the first Dirichlet eigenvalue of Δ provided (4.2) has a positive solution with $q = \mathcal{R}/8$. Hence we may have a result similar to Theorem 4.1 without assuming $\beta < \Lambda$, provided (4.2) has a positive solution with $q = \mathcal{R}/8$.

By Theorem 4.1 and Lemma 4.2, we have:

Corollary 4.2. *With the same assumptions and notations as in Theorem 4.1.*

(a) Suppose $\beta = \left(\int_{\Omega} \left(\frac{\mathcal{R}_-}{8} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} < \frac{1}{2}\Lambda$, then

$$(4.18) \quad \mathfrak{m}_{\text{BY}}(\Omega) \geq \frac{\Lambda - 2\beta}{32(\Lambda + \delta - \beta)} \left(\int_{\Omega} \mathcal{R}_+ - \frac{\Lambda + 2\delta}{\Lambda - 2\beta} \int_{\Omega} \mathcal{R}_- \right)$$

where $\delta = \left(\int_{\Omega} \left(\frac{\mathcal{R}_+}{8} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}}$. In particular, $\mathfrak{m}_{\text{BY}}(\Omega) \geq 0$ if

$$\int_{\Omega} \mathcal{R}_+ \geq \frac{\Lambda + 2\delta}{\Lambda - 2\beta} \int_{\Omega} \mathcal{R}_-.$$

(b) Suppose $8\lambda + \mathcal{R} > 0$ where λ is the first Dirichlet eigenvalue for the Laplacian of Ω , then

$$(4.19) \quad \mathfrak{m}_{\text{BY}}(\Omega) \geq \frac{\lambda}{4} \int_{\Omega} \frac{\mathcal{R}}{8\lambda + \mathcal{R}}.$$

In particular, if $\int_{\Omega} \frac{\mathcal{R}}{8\lambda + \mathcal{R}} \geq 0$, then $\mathfrak{m}_{\text{BY}}(\Omega) \geq 0$.

Hence if the mean curvature of Σ is positive, then the Brown-York mass of Ω is still nonnegative if the scalar curvature of Ω is not too negative.

Next we consider the case that the scalar curvature is positive but the mean curvature H of Σ is negative somewhere. We want to prove that if H is not too negative then $\mathfrak{m}_{\text{BY}}(\Omega)$ is still nonnegative. Let us first consider a special case that $\mathcal{R} \geq 0$, but the mean curvature of the boundary is only assumed to be nonnegative, we have a simple proof of the nonnegativity of the Brown-York mass.

Proposition 4.1. *Let (Ω, g) be a 3-dimensional compact Riemannian manifold with smooth boundary Σ such that Σ has positive Gauss curvature and nonnegative the mean curvature is only assumed to satisfy $H \geq 0$. Then*

$$\mathfrak{m}_{\text{BY}}(\Omega) \geq 0.$$

In fact $\mathfrak{m}_{\text{BY}}(\Omega)$ can be bounded from below by (4.18) and (4.19) with $\mathcal{R}_- = 0$.

Proof. Suppose $H > 0$ at some point $p \in \Sigma$. Let U be an open neighborhood of p in Σ such that $H \geq a > 0$ in U for some positive constant $a > 0$. Let $1 \geq \varphi \geq 0$ be smooth function on Σ with compact support in U such that $\varphi(p) > 0$. For any $\epsilon > 0$, Let v_{ϵ} be the solution of

$$\begin{cases} \Delta v_{\epsilon} - \frac{R}{8} v_{\epsilon} &= 0 \text{ in } \Omega \\ v_{\epsilon} &= 1 - \epsilon \varphi \text{ on } \Sigma. \end{cases}$$

Then $1 > v_\epsilon > 1 - \epsilon$ by maximum principle. Let $g_\epsilon = v_\epsilon^4 g$. Then (M, g_ϵ) has zero scalar curvature, such that the mean curvature of Σ is positive provided $\epsilon > 0$ is small enough, because $v_\epsilon \rightarrow 1$ in C^2 and v_ϵ is not constant. The Gauss curvature of Σ is also positive. Hence

$$\int_{\Sigma} (H_0^\epsilon - H^\epsilon) d\sigma_\epsilon \geq 0$$

where H_0^ϵ, H^ϵ are the mean curvatures of Σ when embedded in \mathbb{R}^3 and in Ω respectively, $d\sigma_\epsilon$ is the area element of with respect to g_ϵ . Let $\epsilon \rightarrow 0$ and use Corollary 4.2, we conclude the proposition is true.

Suppose $H \equiv 0$. If $\mathcal{R} \equiv 0$, then it is obvious the proposition is true. If $\mathcal{R} > 0$ somewhere, then the solution of

$$\begin{cases} \Delta u - \frac{\mathcal{R}}{8}u &= 0 \text{ in } \Omega \\ u &= 1 \text{ on } \Sigma. \end{cases}$$

satisfies $\frac{\partial u}{\partial \nu} > 0$ by the strong maximum principle. In this case, the result follows as in Corollary 4.2.

For a more general case, as before let $s_0 > 0$ be such that $d(x, \partial\Omega)$ is smooth in $\Omega_{s_0} = \{x \mid 0 < d(x, \partial\Omega) \leq s_0\}$. We have the following:

Theorem 4.2. *Let (Ω^3, g) be a bounded domain with smooth boundary Σ with nonnegative scalar curvature \mathcal{R} such that the Gauss curvature of Σ is positive. Let*

$$\xi = \frac{1}{4} \mathcal{R}_{\min}^{\frac{1}{2}}$$

where $\mathcal{R}_{\min} = \inf_{\Omega_{s_0}} \mathcal{R}$. Let H be the mean curvature of $\{x \mid d(x, \partial\Omega) = s\}$, $0 \leq s \leq s_0$, $H_+ = \max\{H, 0\}$ and $H_{\min} = \min_{\partial\Omega} H$. Suppose

- (i) $\xi \geq H_+ \tanh(\xi s_0)$ in Ω_{s_0} ; and
- (ii) $\xi \tanh(\xi s_0) \geq -4H_{\min}$.

Then

$$\mathfrak{m}_{\text{BY}}(\Omega) \geq \frac{1}{4} |\Sigma| \xi \tanh(\xi s_0).$$

In particular, $\mathfrak{m}_{\text{BY}}(\Omega)$ is bounded below by a nonnegative constant depending on \mathcal{R}_{\min} , H_+ , H_{\min} , $|\Sigma|$ and s_0 .

Remark 4.2. *It is easy to see that if $\mathcal{R} \geq 0$ and $H \geq 0$, then the conditions in the theorem will be satisfied. Also, the theorem says that if $\mathcal{R} > 0$, then the Brown-York mass of Ω is still nonnegative provided that mean curvature of its boundary is not very negative.*

Proof. Since $\Delta - \mathcal{R}/8$ has positive first Dirichlet eigenvalue, (4.2) has a unique positive solution u with $q = \mathcal{R}/8$. Note that $0 < u \leq 1$. Let ξ be as in the assumptions of the theorem and let

$$\phi(s) = \cosh(\xi s) - \tanh(\xi s_0) \sinh(\xi s),$$

$0 \leq s \leq s_0$. Then on $0 \leq s \leq s_0$, $\phi > 0$ because $\tanh \xi s \tanh(\xi s_0) < 1$. $\phi(0) = 1$,

$$(4.20) \quad \frac{d^2 \phi}{ds^2} = \xi^2 \phi,$$

$$(4.21) \quad \frac{d\phi}{ds} = \xi (\sinh(\xi s) - \tanh(\xi s_0) \cosh(\xi s))$$

and so $\frac{d\phi}{ds} < 0$ in $0 \leq s < s_0$ and $\frac{d\phi}{ds} = 0$ at $s = s_0$.

Define $f(x) = \phi(d(x, \partial\Omega))$ for $x \in \Omega_{s_0}$, and $f(x) = \phi(s_0)$ for $x \in \Omega \setminus \Omega_{s_0}$. Then f is Lipschitz in Ω and in Ω_{s_0}

$$(4.22) \quad \begin{aligned} \Delta f - \frac{\mathcal{R}}{8} f &= \phi'' - H\phi' - \frac{\mathcal{R}}{8} \phi \\ &\leq \left(\xi^2 - \frac{\mathcal{R}_{\min}}{8} \right) \phi + H_+ \xi (\tanh(\xi s_0) \cosh(\xi s) - \sinh(\xi s)) \\ &\leq \xi \cosh(\xi s) [-\xi (1 - \tanh(\xi s_0) \tanh(\xi s)) + H_+ (\tanh(\xi s_0) - \tanh(\xi s))] \end{aligned}$$

where we have used the fact that $\mathcal{R}_{\min} = 16\xi^2$, $\phi > 0$ and $\phi' \leq 0$ in $0 \leq s \leq s_0$.

On the other hand,

$$(4.23) \quad \begin{aligned} \xi (1 - \tanh(\xi s_0) \tanh(\xi s)) &\geq H_+ (\tanh(\xi s_0) - \tanh(\xi s)) \\ \iff \xi - H_+ \tanh(\xi s_0) &\geq (\xi \tanh(\xi s_0) - H_+) \tanh(\xi s). \end{aligned}$$

Since

$$\xi + H_+ \geq (\xi + H_+) \tanh(\xi s_0),$$

which implies

$$\xi - H_+ \tanh(\xi s_0) \geq \xi \tanh(\xi s_0) - H_+,$$

we have

$$(4.24) \quad \xi - H_+ \tanh(\xi s_0) \geq (\xi \tanh(\xi s_0) - H_+) \tanh(\xi s)$$

if $\xi \tanh \frac{\xi s_0}{2} - \alpha \geq 0$. The above inequality is obvious true if $\xi \tanh \frac{\xi s_0}{2} - \alpha < 0$ because $\xi \geq H_+ \tanh(\xi s_0)$ by condition (i) in the assumptions. From (4.22)–(4.24), we conclude that

$$(2.25) \quad \Delta f - \frac{\mathcal{R}}{8} f \leq 0$$

in Ω_{s_0} . Since $\mathcal{R} \geq 0$, (2.25) is also true in the interior of $\Omega \setminus \Omega_{s_0}$. Since $\phi'(s_0) = 0$, it is easy to see that f satisfies (2.25) weakly in Ω . By the maximum principle and by the fact that $f = 1$ on $\partial\Omega$, we conclude that $u \leq f$ in Ω . Hence

$$(4.26) \quad \frac{\partial u}{\partial \nu} \geq \frac{\partial f}{\partial \nu} = \xi \tanh(\xi s_0)$$

on $\partial\Omega$, where ν is the unit outward normal of $\partial\Omega$.

Consider the metric $g_1 = u^4 g$, then the scalar curvature of g_1 is zero and the mean curvature \overline{H} with respect to g_1 of $\partial\Omega$ satisfies

$$\overline{H} = H + \frac{1}{4} \frac{\partial u}{\partial \nu} \geq H_{\min} + \frac{1}{4} \xi \tanh(\xi s_0) \geq 0$$

by condition (ii) in the assumptions of the theorem. Since $u = 1$ on the boundary, the induced metric on Σ is the same as before and so the Gauss curvature of Σ is positive. By Proposition 4.1, we have

$$\int_{\Sigma} (H_0 - \overline{H}) d\sigma \geq 0,$$

and hence

$$\int_{\Sigma} (H_0 - H) d\sigma \geq |\Sigma| \cdot \frac{1}{4} \xi \tanh(\xi s_0).$$

From this the theorem follows.

Similar to Corollary 4.2, by Theorem 4.2 and Lemma 4.2, we have:

Corollary 4.3. *With the assumptions and same notations as in Theorem 4.2. Then $\mathfrak{m}_{\text{BY}}(\Omega)$ is bounded below as in (4.18) and (4.19) with $\mathcal{R}_- = 0$.*

Proof. By the proof of the theorem, with the same notations as in the proof of the theorem, we have

$$\mathfrak{m}_{\text{BY}}(\Omega) \geq \frac{1}{4} \int_{\Sigma} \frac{\partial u}{\partial \nu}.$$

The corollary follows as in the proof of Corollary 4.2.

§5 Some examples and applications.

In this section, we will give some examples which are related to results in previous sections. Some of the examples might be well-known.

Example 1: In Corollary 3.1, it is proved that if the negative part of the scalar curvature is small compared with the Sobolev constant and the positive part of the scalar curvature, then the ADM mass of a spin AF manifold is nonnegative. The following example show that if we only assume that the negative part of scalar curvature is small compared with the Sobolev constant, the ADM mass might still be negative.

Let $g_{ij} = u^4 \delta_{ij}$ be a conformal metric on \mathbb{R}^3 . Then the scalar curvature of g is

$$(5.1) \quad \mathcal{R} = -8u^{-5} \Delta_0 u,$$

where Δ_0 is the Euclidean Laplacian.

Let v be a nonconstant smooth function such that $\Delta_0 v \geq 0$ in $B(1) = \{x \in \mathbb{R}^3 \mid |x| < 1\}$, say, and $\Delta_0 v = 0$ outside $B(1)$ such that $v \rightarrow 0$ near infinity. Then $v \leq 0$. We may also assume that $v > -1$. Then $v = -\frac{A}{|x|} + O(|x|^{-2})$ near infinity with $A > 0$. For any $1 > \epsilon > 0$, let $u = 1 + \epsilon v$ and consider the metric $g_{ij} = u^4 \delta_{ij}$. Then the scalar curvature $\mathcal{R} \leq 0$ by (5.1). Moreover, if $M = (\mathbb{R}^3, g)$, then

$$\left(\int_M \mathcal{R}_-^{\frac{3}{2}} dV_g \right)^{\frac{2}{3}} = 8\epsilon \left(\int_M (u^{-5} \Delta_0 v)^{\frac{3}{2}} u^6 dV_e \right)^{\frac{2}{3}}.$$

where dV_e is the Euclidean volume form. If ϵ is small enough, then u is close to 1. Hence $\left(\int_M \mathcal{R}_-^{\frac{3}{2}} dV_g \right)^{\frac{2}{3}}$ can be made arbitrarily small compared with the Sobolev constant Λ of M by letting $\epsilon \rightarrow 0$. But the mass of M is negative.

Example 2: It is easy to see that assumptions of Corollary 3.1 or the assumptions in [ZZ, Theorem 4.1] imply $\int_M \mathcal{R} \geq 0$ for an AF manifold M . However, it is not hard to construct examples of AF metrics g on \mathbb{R}^3 such that $\int_M \mathcal{R} > 0$ but the ADM mass \mathfrak{m}_g of M is negative, where $M = (\mathbb{R}^3, g)$.

Let $g_{ij} = u^4 \delta_{ij}$ be an AF metric on \mathbb{R}^3 with $\mathfrak{m}_g < 0$, for example the metric in the example in [ZZ]. Let $v \geq 0$ be a smooth function with support in $B(1) = \{|x| < 1\}$ and $v \not\equiv 0$. Define $\varphi = 1 + av$ where $a > 0$. Consider the metric $\tilde{g}_{ij} = (\varphi u)^4 \delta_{ij}$. Then $\varphi u = u$ outside $B(1)$. Hence $\mathfrak{m}_{\tilde{g}} = \mathfrak{m}_g < 0$. The scalar curvature $\mathcal{R}_{\tilde{g}}$ is given by

$$\mathcal{R}_{\tilde{g}} = -(\varphi u)^{-5} \Delta_0(\varphi u).$$

Hence

$$\begin{aligned} \int_{\mathbb{R}^3} \mathcal{R}_{\tilde{g}} dV_{\tilde{g}} &= \int_{|x|>1} \mathcal{R}_g dV_g - \int_{|x|<1} \varphi u \Delta_0(\varphi u) dV_e \\ &= \int_{|x|>1} \mathcal{R}_g dV_g + \int_{|x|<1} |\nabla_0(\varphi u)|^2 dV_e - \int_{|x|=1} u \frac{\partial u}{\partial r} \\ &= \int_{|x|>1} \mathcal{R}_g dV_g - \int_{|x|=1} u \frac{\partial u}{\partial r} + \int_{|x|<1} |\nabla_0 u + a \nabla(vu)|^2 dV_e \\ &\geq \int_{|x|>1} \mathcal{R}_g dV_g - \int_{|x|=1} u \frac{\partial u}{\partial r} - \int_{|x|<1} |\nabla_0 u|^2 + \frac{1}{2} a^2 \int_{|x|<1} |\nabla_0(vu)|^2 dV_e. \end{aligned}$$

Choose v so that vu is not constant in $|x| < 1$ and choose a large enough, we have $\int_{\mathbb{R}^3} \mathcal{R}_{\tilde{g}} dV_{\tilde{g}} > 0$.

In [B1, Theorem 5.2], it was proved that if a metric g is close to the Euclidean metric of \mathbb{R}^3 and if $\int_{\mathbb{R}^3} \mathcal{R}_g dV_e \geq 0$, then $\mathfrak{m}_g \geq 0$. On the other hand, we have the following observation.

Proposition 5.1. *Suppose $g_{ij} = u^4 \delta_{ij}$ is an AF metric on \mathbb{R}^3 . Then $\mathfrak{m}_g \leq C \int_M \mathcal{R} dV_g$ for some absolute constant $C > 0$. Equality holds if and only if g is Euclidean. In particular, if $\mathfrak{m}_g \geq 0$, then $\int_{\mathbb{R}^3} \mathcal{R} dV_g \geq 0$.*

Proof. Let $v = 1 - u$. Then $v = \frac{A}{r} + O(r^{-2})$, and $|\partial v| = O(r^{-2})$ etc. Then the mass of g is given by

$$(5.2) \quad C\mathfrak{m}_g = - \int_{\partial B(\infty)} \frac{\partial u}{\partial r} = - \int_{\mathbb{R}^3} \Delta_0 u dV_e$$

for some absolute constant $C > 0$. Note that

$$(5.3) \quad \begin{aligned} \int_M \frac{\mathcal{R}}{8} dV_g &= - \int_{\mathbb{R}^3} (u^{-5} \Delta_0 u) u^6 dV_e \\ &= - \int_{\mathbb{R}^3} u \Delta_0 u dV_e \end{aligned}$$

By (5.2) and (5.3), we have

$$\begin{aligned} C\mathfrak{m}_g - \int_M \frac{\mathcal{R}}{8} dV_g &= \int_{\mathbb{R}^3} (-\Delta_0 u + u \Delta_0 u) dV_e \\ &= \int_{\mathbb{R}^3} (-1 + u) \Delta_0 u dV_e \\ &= \int_{\mathbb{R}^3} v \Delta_0 v dV_e \\ &= - \int_{\mathbb{R}^3} |\nabla_0 v|^2 dV_s \\ &\leq 0 \end{aligned}$$

because $v = O(r^{-1})$ and $|\nabla_0 v| = O(r^{-2})$. From this it is easy to see the proposition is true.

Example 3: There are examples of AF metrics defined on \mathbb{R}^3 with zero scalar curvature and positive ADM mass. In fact, Miao [M] constructed an AF metric on \mathbb{R}^3 which is scalar flat, conformally flat outside a compact set and contains a horizon. In particular, the mass is positive.

We may also construct the scalar flat but nonflat AF metrics in the following way. Take a metric g on \mathbb{S}^3 so that (\mathbb{S}^3, g) is not conformal to the standard metric and has positive Yamabe invariant. Take a point p in \mathbb{S}^3 and consider the metric $u^4 g$ on $\mathbb{S}^3 \setminus \{p\}$, where u is the Green's function for the conformal Laplacian with pole at p , which exists and positive by [LP, §6]. Then the manifold $M = (\mathbb{S}^3 \setminus \{p\}, u^4 g)$ is AF, scalar flat, and M is diffeomorphic to \mathbb{R}^3 . Note that M has positive mass and M is not conformal to \mathbb{R}^3 , see [LP, §11]. To construct g , we can perturb the standard metric in some neighborhood of a point p so that it is not conformally

flat in that neighborhood. One may assume the perturbation is small so that the scalar curvature is still positive. Then the Yamabe invariant of the metric must be nonnegative by the definition of the Yamabe functional. It must be positive, otherwise we can find a positive solution of conformal Laplacian. The solution must be constant by the strong maximum principle, which is impossible because the scalar curvature of g is positive. The metric is not conformal to the standard metric because it is not locally conformally flat. So we are done.

Example 4: From Example 3, we can construct AF metrics g on \mathbb{R}^3 so that $\mathcal{R} \leq 0$, $\mathcal{R} \not\equiv 0$ and $\mathfrak{m}_g > 0$.

To do this, let g be an AF metric on \mathbb{R}^3 defined in Example 3 so that $\mathcal{R}_g \equiv 0$ and $\mathfrak{m}_g > 0$. Let v be a nonconstant bounded subharmonic function with respect to g such that v is harmonic outside a compact set so that $v \rightarrow 0$ near infinity. Then $v \sim a/r$ near infinity for some constant a . Let $u = 1 + \epsilon v$ where $\epsilon > 0$ is small and let $\tilde{g} = u^4 g$. Then \tilde{g} is AF metric on \mathbb{R}^3 and there is an absolute constant $C > 0$ such that

$$\mathfrak{m}_{\tilde{g}} = -C \int_{\mathbb{S}_\infty} \frac{\partial u}{\partial r} + \mathfrak{m}_g = -C\epsilon \int_{\mathbb{S}_\infty} \frac{\partial v}{\partial r} + \mathfrak{m}_g.$$

Hence $\mathfrak{m}_{\tilde{g}} > 0$ if ϵ is small enough. On the other hand, since $\mathcal{R}_g \equiv 0$,

$$\mathcal{R}_{\tilde{g}} = -u^{-5} \Delta_g u = -\epsilon u^{-5} \Delta_g v \leq 0$$

and $\mathcal{R}_{\tilde{g}} < 0$ somewhere because $\Delta_g v > 0$ somewhere. In particular, $\int_{\mathbb{R}^3} \mathcal{R}_{\tilde{g}} dV_{\tilde{g}} < 0$.

Applications: We now discuss some relations of the results in [ST1] and [H] and the classical Minkowski's inequalities for convex bodies in \mathbb{R}^3 .

It was proved by Herzlich [H, Proposition 2.1], that if (M, g) is an AF 3-dimensional manifold with nonnegative scalar curvature with an inner boundary Σ which is homeomorphic to \mathbb{S}^2 whose mean curvature with respect to the inner normal satisfies

$$(5.4) \quad H \leq 4 \sqrt{\frac{\pi}{A(\Sigma)}}$$

where $A(\Sigma)$ is the area of Σ , then the mass of M is nonnegative. By convention the mean curvature of $\mathbb{R}^3 \setminus B(1)$ in the Euclidean space is 2. On the other hand, the well-known Minkowski's inequalities for convex bodies in \mathbb{R}^3 state that if Σ is a compact convex surface in \mathbb{R}^3 , then

$$(5.5) \quad \left(\int_{\Sigma} H_0 \right)^2 \geq 16\pi A,$$

and

$$(5.6) \quad \frac{4A^4}{9V^2} \geq \left(\int_{\Sigma} H_0 \right)^2$$

where H_0 is the mean curvature of Σ in \mathbb{R}^3 , A is the area of Σ and V is the volume of the region bounded by Σ , see [BG, p. 438]. Moreover equality holds either in (5.5) or (5.6) if and only if Σ is a standard sphere.

Using the results in [H,ST1], one can derive (5.5). In fact, using the method of [ST1], one can find an AF metric g on the exterior \mathbb{R}_Σ^3 of Σ in \mathbb{R}^3 with zero scalar curvature such that Σ has constant mean curvature $H \equiv 4\sqrt{\frac{\pi}{A(\Sigma)}}$. Note that by the construction in [ST1], the metric g when restricted on Σ is the same as the restriction of the Euclidean metric. By [H], $\mathfrak{m}_g \geq 0$. Hence by the result of [ST1],

$$\int_{\Sigma} (H_0 - H) \geq 0.$$

This implies (5.5).

We can also use (5.6) and the result of [ST1] to prove that condition (5.4) is sharp in the result of [H] mentioned above in the following sense. Given any $\epsilon > 0$, we can find a manifold with boundary satisfying all the conditions of Proposition 2.1 in [H] except that

$$H \geq 4\sqrt{\frac{\pi}{A(\Sigma)}} + \epsilon$$

and the mass of the manifold is negative. To construct such an example, note that equalities hold in (5.5) and (5.6) if and only if Σ is the standard sphere. Hence for any $\epsilon > 0$, we can perturb the standard sphere so that it is still strictly convex, but

$$\frac{4A^4}{9V^2} < 16\pi A + \epsilon.$$

Now we find the metric g in the exterior of Σ as before with initial mean curvature H such that

$$\left(\int_{\Sigma} H \right)^2 = 16\pi A + \epsilon.$$

If the mass is nonnegative, then we have

$$\left(\int_{\Sigma} H_0 \right)^2 \geq \left(\int_{\Sigma} H \right)^2 = 16\pi A + \epsilon > \frac{4A^4}{9V^2}$$

by [ST1], which is impossible because of (5.6).

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